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Semi-infinite TASEP with a Complex Boundary Mechanism

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Abstract We consider a totally asymmetric exclusion process on the positive half-line. When particles enter the system according to a Poisson source, Liggett has computed all the limit distributions when the initial distribution has an asymptotic density. In this paper we consider systems for which particles enter according to a complex mechanism depending on the current configuration in a finite neighborhood of the origin. For this kind of models, we prove a strong law of large numbers for the number of particles which have entered the system at a given time. Our main tool is a new representation of the model as a multi-type particle system with infinitely many particle types.

Keywords Particle systems · Exclusion process · Coupling methods

1 Introduction

The simple exclusion process $\eta_. = (\eta_t)_{t\geq 0}$ on a countable space S, with random walk kernel $p(.)$, is a continuous time Markov process on $X := \{0,1\}^S$. For a configuration $\eta \in X$, we say that the site x is *occupied* (by a particle) if $\eta(x) = 1$, and is *empty* if $\eta(x) = 0$. A particle "tries" to move from an occupied site x to an empty site y at rate $p(x,y)$, or in an equivalent way, waits for an exponential time of parameter 1 and then chooses a site y randomly with probability $p(x,y)$ and "tries" to jump on y. If the site y is already occupied, the jump is cancelled and the particle stays at x, otherwise it jumps to y. In this way, there is always at most one particle at any given site. Formally, the exclusion process $\eta_.$ is defined as the Feller process with generator

$$\Omega f(\eta) := \sum_{x,y\in S} p(x,y)\eta(x)(1-\eta(y))[f(\eta_{x,y}) - f(\eta)], \tag{1}$$

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for all cylindrical functions f, where

$$\eta_{x,y}(z) := \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \\ \eta(z) & \text{otherwise.} \end{cases}$$

A natural question is to describe the set of invariant probability measures $\mathcal{I}$, which is the set of probability measures μ on S such that, if $\eta_0 \sim \mu$ then for all $t \geq 0$, $\eta_t \sim \mu$. These measures are characterized by the equations:

$$\int \Omega f \, \mu(d\eta) = 0,$$

for any cylindrical functions f (see e.g. [5] for a review). We denote by $\mathcal{I}_e$ the set of extreme points of $\mathcal{I}$. In the case $S = \mathbb{Z}$, the set of extremal, translation-invariant stationary measures is exactly the set of translation-invariant Bernoulli product measures on $\mathbb{Z}$ (see [4]).

In this paper, we consider the case $S := \mathbb{Z}_+^*$ and $p(x, x+1) := 1$, *i.e.*, the totally asymmetric nearest neighbor case. In $\mathbb{Z}_+^*$, one has to add some boundary mechanism to make the model non trivial. The simplest way to do this is to add a particle reservoir at site 0 with a certain density $\lambda > 0$. This means that a new particle is created at site 1 according to a Poisson process with rate λ when this site is empty. We call the model on $\mathbb{Z}_+^*$ TASEP(λ), and we denote by Ω_λ its generator and by $S_\lambda(t)$ its semi-group:

$$\begin{aligned} \Omega_\lambda f(\eta) &:= \lambda(1 - \eta(1))[f(\eta_1) - f(\eta)] \\ &\quad + \sum_{x=1}^{\infty} \eta(x)(1 - \eta(x+1))[f(\eta_{x,x+1}) - f(\eta)], \end{aligned} \tag{2}$$

for all cylindrical functions f, where

$$\eta_1(z) := \begin{cases} 1 - \eta(1) & \text{if } z = 1, \\ \eta(z) & \text{otherwise.} \end{cases}$$

In (2) we see two parts for the generator: one is due to the boundary mechanism and we will call it the *boundary part*; the other one, which has the form given by (1) for $S = \mathbb{Z}_+^*$, is due to the exclusion process and we will call it the *bulk part*.

Let us introduce some notation. In the following, we denote by ν^λ the product measure on $\{0, 1\}^{\mathbb{Z}_+^*}$ with density λ and by θ the shift. θ acts on configurations $\eta \in X$ by

$$\theta\eta(x) := \eta(x+1), \quad \forall x \in \mathbb{Z}_+^*,$$

on functions $f : X \to \mathbb{R}$ by

$$\theta f(\eta) := f(\theta\eta), \quad \forall \eta \in X,$$

and on measures μ on X by

$$\int f d\theta\mu := \int \theta f d\mu, \quad \forall f \in L^1(\mu).$$

For a measure μ on S and $f \in L^1(\mu)$, we will denote $\langle f \rangle_\mu := \int f d\mu$.

We are interested in the asymptotic behavior of the distribution when t goes to infinity. For this model, we have a good understanding about what happens at equilibrium. Indeed, Liggett has shown in [3] the following ergodic theorem, which gives the limit measure for an initial measure with a product form and an asymptotic density:

Theorem 1.1 (Liggett [3]) *Let π be a product measure on $\mathbb{Z}_+^*$ for which $\rho := \lim_{x\to\infty}\langle \eta(x)\rangle_\pi$ exists.*

$$\textit{If } \lambda \geq \frac{1}{2} \quad \textit{then } \lim_{t\to\infty} \pi S_\lambda(t) = \begin{cases} \mu_\rho^\lambda, & \textit{if } \rho \geq \frac{1}{2} \ (\textit{bulk dominated}),\\ \mu_{\frac{1}{2}}^\lambda, & \textit{if } \rho \leq \frac{1}{2} \ (\textit{maximum current}).\end{cases}$$

$$\textit{If } \lambda \leq \frac{1}{2} \quad \textit{then } \lim_{t\to\infty} \pi S_\lambda(t) = \begin{cases} \mu_\rho^\lambda, & \textit{if } \rho > 1-\lambda \ (\textit{bulk dominated}),\\ \nu^\lambda, & \textit{if } \rho \leq 1-\lambda \ (\textit{boundary dominated}),\end{cases}$$

where the μ_ρ^λ's, for $\rho \geq \frac{1}{2}$, are stationary measures and asymptotically product with density ρ, i.e., $\lim_{x\to\infty} \theta^x \mu_\rho^\lambda = \nu^\rho$ *(in a weak sense with test functions $f \in C(X,\mathbb{R})$). We also have $\mu_\lambda^\lambda = \nu^\lambda$.*

To describe the set of invariant probability measures in the cases $S = \mathbb{Z}$ and $S = \mathbb{Z}_+^*$, Liggett uses that the Bernoulli product measures are invariant and for these measures one can make explicit computations. In this paper, we study Markov processes with no invariant product measure. We consider a TASEP on $\mathbb{Z}_+^*$ for which the boundary rate depends on the current configuration. We limit ourselves to finite range boundary mechanisms, *i.e.*, systems for which there exist some $R \in \mathbb{Z}_+^*$ such that the boundary part of the generator vanishes on every cylindrical function with support in $\{R+1,\ldots\}$. This idea was first introduced by Großkinsky in Chap. 3 of his PhD Thesis [1] where he defines the following Feller process:

$$\Omega f(\eta) := \sum_{x\in\mathbb{Z}_+^*} \eta(x)\,(1-\eta(x+1))\left[f(\eta_{x,x+1}) - f(\eta)\right] + \sum_{\xi\in X_R} d_{\eta_{|S_R},\xi}\left[f(\xi\cup\eta_{|^cS_R}) - f(\eta)\right], \tag{3}$$

for all cylindrical functions f where $S_R := \{1,\ldots,R\}$, $X_R := \{0,1\}^{S_R}$, $\eta_{|S_R}$ and $\eta_{|^cS_R}$ are the configuration η restricted to S_R and ${}^cS_R = \mathbb{Z}_+^*\backslash S_R$ respectively, $\xi\cup\eta_{|^cS_R}$ is the natural concatenation of configurations on S_R and on ${}^cS_R$, and $(d_{\xi,\xi'})_{\xi,\xi'\in X_R}$ are non-negative rates.

Assuming the existence of an invariant measure which is product outside of the box $\{1,\ldots,R\}$ with a non-trivial density leads to relations which the boundary rates have to satisfy—we will refer to such models as *almost classic*. These are still within the reach of Theorem 1.1, at least for suitable choices of λ and ρ. From now on, we will assume that at least one of these relations is not satisfied by our boundary mechanism.

Remark The reason for which we only treat the finite range case is that when we are not in this case, pathological things can occur. For example, consider the following dynamic with a non-local boundary mechanism. Define the asymptotic density of a configuration $\eta\in X$ by $\rho(\eta) := \liminf_{x\to\infty}\frac{1}{x}\sum_{i=1}^x \eta(i)$; we consider now a TASEP on $\mathbb{Z}_+^*$ for which the rate of apparition of a particle in site 1 is $\rho(\eta)$ where η is the current configuration. More formally, the boundary part of the generator is

$$\rho(\eta)(1-\eta(1))[f(\eta_1) - f(\eta)].$$

In this example, every mixture of Bernoulli product measures is invariant for the process. Admittedly this case is too extreme; probably, suitable decay of dependency would create a behavior similar to the finite dependency case.

For this generalized boundary mechanism, we will not have an exact solution as for the TASEP(λ). Indeed, one can check that this process is not almost classic and then does not have any invariant measure which is of product form. Our approach is to study the number of particles which have entered the system by time t. We will see that it grows linearly with an almost sure speed equal to the stationary current $j_\infty := \mu_\infty\{\eta \in X : \eta(1) = 1, \eta(2) = 0\}$ for an invariant measure μ_∞. Define ρ_∞ as the root of $\rho(1-\rho) = j_\infty$ in $[0, 1/2[$. We believe that the process has a stationary measure which is asymptotically product with density ρ_∞; but we are still unable to prove it.

The rest of the paper is organized as follow: in Sect. 2 we give a construction of the process defined above using a graphical representation similar to that introduced by Harris [2]. We also introduce the basic coupling technique which is the main tool used in the paper; in Sect. 3 we give some general results on the asymptotic behavior of the TASEP with complex boundary mechanism. In particular, we show that, starting from the empty configuration, the process converges in distribution to an invariant ergodic measure μ_∞; finally, in Sect. 4 we study a particular example: take a TASEP(λ) on $\mathbb{Z}_+^*$ and add a source (independent of everything) with density $\epsilon > 0$ which is activated only when site 2 is occupied. For this model, let N_t be the number of particles which have entered the system between 0 and t. Then the main result of this paper is the following strong law of large numbers:

Theorem 1.2 *Let* $0 \le \lambda < \frac{1}{2}$, $\epsilon > 0$. *Then starting from* μ_∞,

$$\lim_{t\to\infty} \frac{N_t}{t} = \lambda(1-\lambda) + \lambda(1-\lambda)p(\lambda)\epsilon + o(\epsilon),$$

with probability one, *where* $p(\lambda)$ *is a positive constant* (*depending only on* λ) *for which we give a natural probabilistic interpretation*.

It should be noted that this particular choice of boundary mechanism is rather arbitrary, and that our method is robust enough to be used in a much larger generality. However, the notations which would be needed would be much more tedious, while providing very little additional insight into the model—so we choose to limit ourselves to one representative case.

2 The Harris Construction

We will use the method developed by Harris [2] to construct our process. Let

$$\mathcal{N} := \left(\mathcal{N}_x, \mathcal{N}_{\eta,\eta'}; x \in \mathbb{Z}_+^*, \eta, \eta' \in \{0,1\}^{\{1,\dots,R\}}\right),$$

be a family of independent Poisson point processes on $\mathbb{R}_+^*$ constructed on the same probability space $(\Gamma, \mathcal{F}, \mathbf{P})$, such that the rate of the processes indexed by $\mathbb{Z}_+$ is 1 and the rate of the process indexed by (η, η') is $d_{\eta,\eta'} \ge 0$. By discarding a $\mathbf{P}$-null set, we may assume that

each Poisson point process in $\mathcal{N}$ has only finitely many jump times in every bounded interval $[0, T]$, and no two distinct processes have a jump in common. (4)

We denote

$$\mathcal{N}_0 := \bigcup_{\eta,\eta' \in \{0,1\}^{\{1,\dots,R\}}} \mathcal{N}_{\eta,\eta'}.$$

Fix $T > 0$ and $\eta \in X$. The process $(\eta_t)_{0\le t\le T}$ starting from η is now constructed as follows. Consider the following subgraph of $\mathbb{Z}_+$:

$$\mathcal{G}_T := \Big\{\{x, x+1\} : x \ge R, \mathcal{N}_x \cap [0,T] \neq \emptyset\Big\}$$
$$\cup \Big\{\{x, x+1\} : x \in \{0, \dots, R-1\}\Big\}.$$

It is easy to see that every connected component of $\mathcal{G}_T$ is almost surely finite. Let Γ_0 be the subset of Γ such that (4) and the above condition hold for all $T \ge 0$. Then we have $\Gamma_0 \in \mathcal{F}$ and $\mathbf{P}[\Gamma_0] = 1$. We consider now only $\omega \in \Gamma_0$. For every connected component $\mathcal{C}$ of $\mathcal{G}_T$, the set $(\bigcup_{x\in\mathcal{C}} \mathcal{N}_x) \cap [0,T]$ is finite so its elements can be ordered chronologically $\tau_1 < \cdots < \tau_n$ and we need only to describe the action of each of them. We start with the configuration η:

$$\eta_t(x) := \eta(x)$$

for all $x \in \mathcal{C}$ and $0 \le t < \tau_1$.

Suppose that the process is constructed on $\mathcal{C}$ for $0 \le t < \tau_k$ and $k \in \{1, \dots, n\}$. Then:

- if $\tau_k \in \mathcal{N}_{\xi,\xi'}$ and if $\eta_{\tau_k^-|S_R} = \xi$ then $\eta_{\tau_k|S_R} := \xi'$ and $\eta_{\tau_k}(x) := \eta_{\tau_k^-}(x)$ for all $x \in \mathcal{C}\backslash S_R$,
- if $\tau_k \in \mathcal{N}_{\xi,\xi'}$ and if $\eta_{\tau_k^-|S_R} \neq \xi$ then $\eta_{\tau_k}(x) := \eta_{\tau_k^-}(x)$ for all $x \in \mathcal{C}$,
- if $\tau_k \in \mathcal{N}_x$ and $\eta_{\tau_k^-}(x)(1 - \eta_{\tau_k^-}(x+1)) = 1$ then $\eta_{\tau_k} := (\eta_{\tau_k^-})_{x,x+1}$ on $\mathcal{C}$,
- if $\tau_k \in \mathcal{N}_x$ and $\eta_{\tau_k^-}(x)(1 - \eta_{\tau_k^-}(x+1)) \neq 1$ then $\eta_{\tau_k} := \eta_{\tau_k^-}$ on $\mathcal{C}$.

Finally, we put $\eta_t := \eta_{\tau_k}$ on $\mathcal{C}$ for $\tau_k \le t < \tau_{k+1}$ if $k < n$ and for $\tau_n \le t \le T$ if $k = n$. We make the same construction on every connected component of $\mathcal{G}_T$ and then let T go to infinity to get the process $(\eta_t)_{t\ge 0}$ for every $\omega \in \Gamma_0$.

The usefulness of such a construction is that, using the same Harris process, we can construct two or more realizations of the process on the same probability space starting from different initial configurations. We will refer to this coupling as the *basic coupling*.

3 The Attractive Case

Recall the usual definition of *attractiveness* (or *monotonicity*). Define a partial order on X as follow:

$$\eta \le \xi \quad \text{iff} \quad \forall x \in \mathbb{Z}_+^*,\ \eta(x) \le \xi(x).$$

A function f on X is called *increasing* if $\eta \le \xi$ implies $f(\eta) \le f(\xi)$. This leads to the usual definition of the stochastic monotonicity: $\mu_1 \prec \mu_2$ iff $\langle f\rangle_{\mu_1} \le \langle f\rangle_{\mu_2}$ for every increasing function f. We say that a process on X is *attractive* (or *monotone*) if one of the following equivalent statements hold:

for every increasing function f, $S(t)f$ is also increasing for all $t \ge 0$,

and

$$\mu_1 \prec \mu_2 \text{ implies } \mu_1 S(t) \prec \mu_2 S(t) \text{ for all } t \ge 0.$$

In this section, we consider the process with generator (3) and we assume the process attractive.

3.1 The Stationary Measure

Proposition 3.1 *Assume that the process is attractive* (*or monotone*). *We start from the empty configuration and we denote by* μ_t *the distribution of the process at time* t. *Then*, *the process* $(\mu_t)_{t\geq 0}$ *is stochastically increasing and converges to a measure* $\mu_\infty \in \mathcal{I}$, *which is the smallest invariant measure of the dynamic. Furthermore*, $\mu_\infty \in \mathcal{I}_e$ *and* μ_∞ *is ergodic.*

Proof Let $0 \leq s < t$. We have $\delta_0 \prec \mu_{t-s}$, where δ_0 is the measure charging the empty configuration. Thus by monotonicity of the process, we have $\delta_0 S(s) \prec \mu_{t-s} S(s)$, *i.e.*, $\mu_s \prec \mu_t$. Hence, by monotonicity, μ_t converges weakly to an invariant measure μ_∞.

For all $\nu \in \mathcal{I}$, we have $\delta_0 \prec \nu$, which implies that $\mu_t \prec \nu$ for all $t \geq 0$, and then $\mu_\infty \prec \nu$. Assume now that $\mu_\infty = \lambda\nu_1 + (1-\lambda)\nu_2$, with $\nu_1, \nu_2 \in \mathcal{I}$ and $\lambda \in]0, 1[$. We have $\mu_\infty = \lambda\nu_1 + (1-\lambda)\nu_2 \succ \mu_\infty$, thus $\nu_1 = \nu_2 = \mu_\infty$ and μ_∞ is extremal. Finally, by Theorem B52 of [6], μ_∞ is also ergodic. □

Proposition 3.2 $\theta^R \mu_\infty$ *is stochastically dominated by the measure* $\mu^1_{1/2}$ *of Theorem* 1.1.

Proof Define $\mathcal{N}' := (\mathcal{N}'_x, x \in \mathbb{Z}_+)$, where $\mathcal{N}'_x := \mathcal{N}_{x+R}$. Then $\mathcal{N}'$ defines a TASEP (ξ_t) on $\mathbb{Z}^*_+$ with rate 1 of particle apparition in 1. By Theorem 1.1, starting from the empty configuration, the distribution at time t converges to $\mu^1_{1/2}$. In this coupling, we have $\xi_t(x) \geq \eta_t(x+R)$ almost surely for all $t \geq 0$ and $x \geq 1$. Thus the restriction of μ_∞ to $\{R+1, R+2, \ldots\}$ is stochastically dominated by $\mu^1_{1/2}$. □

3.2 Asymptotic Measures

Let us extend the measure μ_∞ to a measure on $\{0,1\}^{\mathbb{Z}}$ by

$$\overline{\mu}_\infty(A) := \mu_\infty\{\eta \in X : \tilde{\eta} \in A\}, \quad \text{where } \tilde{\eta}(x) := \begin{cases} \eta(x) & \text{if } x \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

for all A in the product σ-field of $\{0,1\}^{\mathbb{Z}}$. By a slight abuse of notation, we still denote this measure by μ_∞. Let $\mu^k := \theta^k \mu_\infty$ and consider any weak limit μ^∞ of this sequence; let $k_i \uparrow \infty$ such that:

$$\lim_{i\to\infty} \mu^{k_i} = \mu^\infty.$$

Proposition 3.3 *The measure* μ^∞ *is a translation invariant stationary measure for TASEP on* $\mathbb{Z}$. *Consequently, it is a mixture of Bernoulli product measures*, *i.e.*, *there exists a probability measure* σ *on* $[0,1]$ *such that*

$$\mu^\infty = \int_0^1 \nu^\lambda \sigma(d\lambda).$$

Proof Let Ω^e be the generator of the TASEP on $\mathbb{Z}$. For any cylindrical function $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, let $x \in \mathbb{Z}^*_+$ large enough such that $\operatorname{supp} \theta^x f \subset \{R+1, R+2, \ldots\}$, where $\operatorname{supp} f$ is the support of f. Thus $\theta^x f$ could be considered has a function on $\mathbb{Z}^*_+$ and we can apply the generator Ω to this function. We get $\Omega\theta^y f = \Omega^e\theta^y f$ for all $y \geq x$. But it is easy to see that Ω^e and θ commute, thus we have

$$\int \Omega\theta^y f \mu_\infty(d\eta) = 0 = \int \theta^y \Omega^e f \mu_\infty(d\eta) = \int \Omega^e f \mu^y(d\eta).$$

Hence for i large enough, $\langle \Omega^e f\rangle_{\mu^i} = 0$, which implies that $\langle \Omega^e f\rangle_{\mu^\infty} = 0$. This is true for arbitrary f thus μ^∞ is invariant for the TASEP on $\mathbb{Z}$. We know that for this model we have $\mathcal{I}_e = \{\nu^\lambda, \lambda \in [0,1]\} \cup \{\nu_n, n \in \mathbb{Z}\}$, where $\nu_n = \theta^n \nu_0$ and ν_0 is the Dirac measure of the configuration for which all the sites $x \geq 0$ are occupied and all the sites $x < 0$ are empty (see [4]). Using Proposition 3.2, since $\mu^1_{1/2}$ is asymptotically product with density $\frac{1}{2}$, μ^∞ is stochastically dominated by $\nu^{1/2}$. Thus μ^∞ is translation invariant and is a mixture of Bernoulli product measures. □

3.3 A Strong Law of Large Numbers

Let μ be an invariant and ergodic measure for the process with generator given by (3). Fix ξ_0, ξ and ξ' three configurations on S_R and consider

$$N(t) := \sharp(\mathcal{N}_{\xi,\xi'} \cap I_t),$$

with $I_t := \overline{\{s \in [0,t] : \eta_{s|S_R} = \xi_0\}}$, where we denote by $\overline{A}$ the closure of a set $A \subset \mathbb{R}_+$. We show a strong law of large numbers for $N(t)$ which will be useful in the sequel.

Proposition 3.4 *If η_0 is distributed according to μ and if $\xi' \neq \xi_0$, then almost surely*:

$$\lim_{t\to\infty} \frac{N(t)}{t} = d_{\xi,\xi'}\mu\left\{\eta \in X : \eta_{|S_R} = \xi_0\right\}.$$

Proof Let

$$T_t := \int_0^t \mathbf{1}_{\eta_{s|S_R}=\xi_0} ds,$$

and

$$\psi(t) := \inf\{s \geq 0 : T_s = t\}.$$

Since μ is ergodic, $T_t/t \underset{t\to\infty}{\longrightarrow} \mu\{\eta \in X : \eta_{|S_R} = \xi_0\}$ almost surely. Let $I := \overline{\{t > 0 : \eta_{t|S_R} = \xi_0\}}$. $\psi : \mathbb{R}_+^* \to I$ is a one to one map, since it is increasing, thus we can define $\mathcal{M} := \psi^{-1}(\mathcal{N}_{\xi,\xi'} \cap I)$ and $N'(t) := \sharp(\mathcal{M} \cap]0,t])$ the associated counting process. We have $N'(t) = N(\psi(t))$ almost surely.

Claim: $\mathcal{M}$ is a Poisson point process with parameter $d_{\xi,\xi'}$.

Let $\tilde{\tau}_0 := 0$ and for $i \geq 1$:

$$\tau_i := \inf\{t > \tilde{\tau}_{i-1} : \eta_{t|S_R} = \xi_0\}, \qquad \tilde{\tau}_i := \inf\{t > \tau_i : \eta_{t|S_R} \neq \xi_0\} \quad \text{and} \quad J_i = \left[\tau_i, \tilde{\tau}_i\right].$$

$(\tau_i)_{i\geq 1}$ and $(\tilde{\tau}_i)_{i\geq 1}$ are stopping times for the process $(\mathcal{N} \cap [0,t])$. To prove the claim we need to distinguish two cases.

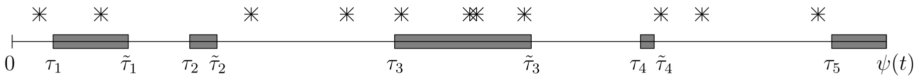

Fig. 1 On the time interval $[0, \psi(t)]$ we see the set $I_{\psi(t)}$ in grey. The total length of the grey part is t. The stars are points of the process $\mathcal{N}_{\xi,\xi'}$. In this example, $N'(t) = 5$

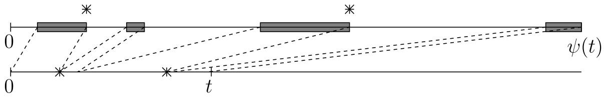

Fig. 2 In the first line, we see the time interval $[0, \psi(t)]$: the set $I_{\psi(t)}$ is in grey; the stars are points of the process $\mathcal{N}_{\xi,\xi'}$. They are always at the end of intervals of $I_{\psi(t)}$ since they change the current configuration. In the second line, we see the Poisson point process $\mathcal{N}_{\xi,\xi'}$ viewed from $I_{\psi(t)}$, *i.e.*, the set $\mathcal{M} \cap\,]0, t]$

Case $\xi \neq \xi_0$: (see Fig. 1) In this case, the points of $\mathcal{N}_{\xi,\xi'} \cap I$ have no effect on the configuration. Hence for each $i \geq 1$, with the strong Markov property, τ_i and the length of J_i are independent of $\mathcal{N}_{\xi,\xi'} \cap [\tau_i, \infty[$. Consequently, conditionally to J_i, $\mathcal{N}_{\xi,\xi'} \cap J_i$ is a Poisson point process with parameter $d_{\xi,\xi'}$. Again with the strong Markov property, $(\mathcal{N}_{\xi,\xi'} \cap J_i)_{i\geq 1}$ are independent conditionally to I. Hence, the claim follows.

Case $\xi = \xi_0$: (see Fig. 2) In this case, each $\mathcal{M}_i := \mathcal{N}_{\xi,\xi'} \cap J_i$ has, almost surely, at most 1 point, thus we have to argue in a different way. For $i \geq 1$, let

$$\sigma_i := \inf \mathcal{N}_{\xi,\xi'} \cap [\tau_i, \infty[$$

and

$$\sigma_i' := \inf \bigcup_{\xi'' \in X_R \setminus \{\xi_0, \xi'\}} (\mathcal{N}_{\xi_0,\xi''} \cap [\tau_i, \infty[) \bigcup_{\substack{x\in\{1,\dots,R\}:\\ \eta_{\tau_i}(x)=1,\eta_{\tau_i}(x+1)=0}} (\mathcal{N}_x \cap [\tau_i, \infty[).$$

The interpretation of σ_i and σ_i' is simple: if $\sigma_i < \sigma_i'$, then the time interval J_i ends with a jump in $\mathcal{N}_{\xi,\xi'}$ and $\mathcal{M}_i$ contains one point ($\mathcal{M}_i = \{\tilde\tau_i\}$); if $\sigma_i > \sigma_i'$, then the time interval J_i ends with an other jump and $\mathcal{M}_i$ is empty. By the strong Markov property, the sequence $(\sigma_i)_{i\geq 1}$ is i.i.d. with distribution exponential with parameter $d_{\xi,\xi'}$. Furthermore, because of the independence of the Poisson point processes in $\mathcal{N}$, $(\sigma_i)_{i\geq 1}$ and $(\sigma_i')_{i\geq 1}$ are independent. By construction, $\inf \mathcal{M} = \sum_{i\leq I} \min(\sigma_i, \sigma_i')$, where $I := \min\{i \geq 1 : \sigma_i < \sigma_i'\}$. Hence, using basic properties of Poisson processes, it is easy to see that $\inf \mathcal{M}$ is an exponential random variable with parameter $d_{\xi,\xi'}$. Finally, using again the strong Markov property, the claim follows.

Then, for every $\epsilon > 0$, $\psi(T_t) \leq t \leq \psi(T_t + \epsilon)$. Since $N(t)$ is non-decreasing, we get $N'(T_t) \leq N(t) \leq N'(T_t + \epsilon)$. Consequently:

$$\frac{N'(T_t)}{T_t}\frac{T_t}{t} \leq \frac{N(t)}{t} \leq \frac{N'(T_t+\epsilon)}{T_t+\epsilon}\frac{T_t+\epsilon}{t}.$$

Since both sides converge to $d_{\xi,\xi'}\mu\{\eta \in X : \eta_{|S_R} = \xi_0\}$ almost surely, it leads to the conclusion. □

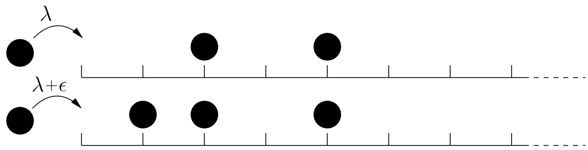

Fig. 3 Particles enter with additional rate ϵ when the site 2 is occupied

4 A Particular Case and the Multi-species Model

In this section, we are interested in a particular case of TASEP with a complex boundary mechanism: let $\lambda, \epsilon > 0$ such that $\lambda + \epsilon < \frac{1}{2}$. Particles are created at site 1 with rate $\lambda + \epsilon\eta(2)$, where η is the current configuration and the bulk dynamic is the one of the TASEP. This model has a generator given by:

$$\Omega f(\eta) := \sum_{x \in \mathbb{Z}_+^*} \eta(x)\,(1-\eta(x+1))\left[f(\eta_{x,x+1}) - f(\eta)\right]$$
$$\times\,(1-\eta(1))(\lambda+\epsilon\eta(2))\left[f(\eta_1) - f(\eta)\right], \tag{5}$$

for all cylindrical functions f on X. As it is explained in the introduction, the choice of the model is rather arbitrary, and the methods that we use are quite robust (at least as long as the system can be dominated by a Bernoulli product measure of intensity lower than 1/2—which is indeed the case here).

In this model, the range of the boundary mechanism is $R = 2$. The hypothesis $\epsilon > 0$ implies that the process is monotone, thus we can define the smallest stationary measure $\mu_\infty = \mu_\infty(\lambda, \epsilon)$ of the model. Using the Harris representation, we can couple this process with $\eta^\lambda_\cdot$, a TASEP(λ), and $\eta^{\lambda+\epsilon}_\cdot$, a TASEP($\lambda+\epsilon$), in such a way that if $\eta^\lambda_0 \le \eta_0 \le \eta^{\lambda+\epsilon}_0$ then for all $t \ge 0$, $\eta^\lambda_t \le \eta_t \le \eta^{\lambda+\epsilon}_t$. This proves that $\nu^\lambda \prec \mu_\infty \prec \nu^{\lambda+\epsilon}$ and then $\nu^\lambda \prec \mu_\infty \prec \nu^{\lambda+\epsilon}$.

4.1 Some Estimates About the Particle Flux

Here we will see another way to see the process with generator given by (5). For any $i \ge 1$, let

$$\mathcal{X}_i := \{\infty, 1, \ldots, i\}^{\mathbb{Z}_+^*}.$$

We define

$$\begin{aligned}\Omega^{(i)} f(\eta) := {} & \lambda \mathbf{1}_{\eta(1)\ge 2}[f(\eta_{1\to 1}) - f(\eta)] \\ & + \sum_{j=2}^{i} \epsilon \mathbf{1}_{\eta(1)\ge j+1}\mathbf{1}_{\eta(2)=j-1}[f(\eta_{j\to 1}) - f(\eta)] \\ & + \epsilon \mathbf{1}_{\eta(1)=\infty}\mathbf{1}_{\eta(2)=i}[f(\eta_{i\to 1}) - f(\eta)] \\ & + \sum_{x=1}^{\infty} \mathbf{1}_{\eta(x+1)>\eta(x)}[f(\eta_{x,x+1}) - f(\eta)],\end{aligned} \tag{6}$$

for all cylindrical function $f : \mathcal{X}_i \to \mathbb{R}$, where

$$\eta_{j\to 1}(x) := \begin{cases} j & \text{if } x = 1, \\ \eta(x) & \text{otherwise,}\end{cases}$$

for $j \in \{1, \ldots, i\}$.

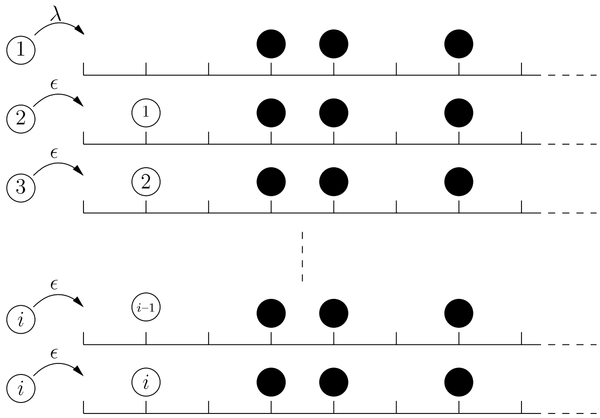

Fig. 4 First class particles enter with rate λ whatever is the configuration in $\{2, 3, \ldots\}$ and second class particles enter with rate ϵ if the site 2 is occupied by a first class particle. Particles in black are indistinguishable particles (their class has no influence on the rate of the source in the current configuration)

We fix $i \geq 2$ for the sequel. The new description is described in Fig. 4 and in the following. We put the particles into a certain number of classes. For a configuration $\eta \in \mathcal{X}_i$ and for a site $x \in \mathbb{Z}_+^*$, the number $\eta(x)$ designates the class of the particle at site x if it exists, *i.e.*, if $\eta(x) \neq \infty$, and is equal to ∞ if the site is empty. We use here another notation for empty sites because it allows us to have a simpler expression for the generator and we can also interpret holes as particles of class infinity. The evolution is the same as before, except that if a particle of the k-th class (or of type k) attempts to jump on a site occupied by a particle of the j-th class (or of type j), then it is not allowed to do so if $k \geq j$, and the particles exchange positions if $k < j$. We say that a particle of class $k \in \{1, 2, \ldots\}$ has priority over all particles of classes greater than k. In this way, a particle of type k behaves as a hole for particles of type $j < k$.

Now we will explain how we affect classes to the particles. First class particles enter the system (at site 1) at rate λ. As they have priority over other particles, they are not affected by them, so the process of first class particles is simply a TASEP(λ) on $\mathbb{Z}_+^*$. Next, particles of class $2 \leq j \leq i-1$ enter the system with rate ϵ, if the site 2 is occupied by a particle of class $j-1$ and with rate 0 otherwise. Finally, particles of class i enter the system with rate ϵ if the site 2 is occupied by a particle of class $i-1$ or i and with rate 0 otherwise. For each configuration of the system, at most 2 types of particles are allowed to enter the system. We can also remark that if we consider the process consisting with particles of class $1, \ldots, i$, then it has the generator given by (5).

In terms of the Harris system, we define $\mathcal{N}$ the collection of the following independent Poisson point processes on $\mathbb{R}_+^*$: let $(\mathcal{N}_x, x \geq 1)$ be Poisson point processes of rate 1; let $(\mathcal{N}_j^b, j \geq 1)$ be Poisson point processes of rate λ for $\mathcal{N}_1^b$ and of rate ϵ for the others. In the sequel, we consider holes as particles of class infinity. The mechanism is then the following: if $t \geq 0$ is a jump time of $\mathcal{N}_x$ and if at time t^- we have $\eta(x+1) > \eta(x)$ (*i.e.*, the particle at x has higher priority than the one at $x+1$), then the particles at x and $x+1$ swap; if $t \geq 0$ is a jump time of $\mathcal{N}_1^b$ and if at time t^- we have $\eta(1) \geq 2$, then a first class particle appears at site 1; if $t \geq 0$ is a jump time of $\mathcal{N}_j^b$ with $2 \leq j \leq i-1$ and if at time t^- we have $\eta(1) \geq j+1$ and $\eta(2) = j-1$, then a j-particle appears at site 1; finally, if $t \geq 0$ is a jump time of $\mathcal{N}_i^b$ and if at time t^- we have $\eta(1) = \infty$ and $\eta(2) \in \{i-1, i\}$, then an i-particle appears at site 1.

We denote by $S^{(i)}(t)$ the semi-group corresponding to the generator $\Omega^{(i)}$ and by $(\eta_t^{(j)})_{t \geq 0}$ the process of the j-th class particles for $j = 1, \ldots, i$, *i.e.*, $\eta_t^{(j)}(x) := \mathbf{1}_{\eta_t(x)=j}$. The process is attractive, thus we can define $\mu_\infty^{(i)}$ as the weak limit of $\delta_0 S^{(i)}(t)$. As in Proposition 3.1, this measure is extremal, ergodic and the smallest invariant measure of the system. For all $1 \le j \le i$, we denote $\bar{\eta}_t^{(j)} := \sum_{k=1}^{j} \eta_t^{(k)}$. Remark that the process $(\bar{\eta}_t^{(i)})_{t\ge 0}$ is exactly the process that we want to study, *i.e.*, it has the generator given by (5). Furthermore, for all $j \ge 1$ the distribution of the process $(\bar{\eta}_t^{(j)})_{t\ge 0}$ is the same for all $i \ge j+1$, *i.e.*, the generator of this process is independent of i since changing the value of i is equivalent to adding or removing some particles with lower priority.

In order to compare the processes $(\bar{\eta}_t^{(i-1)})_{t\ge 0}$ and $(\bar{\eta}_t^{(i)})_{t\ge 0}$, we need to control the number of particles of a given type in the system at a given time. Let $N_t^{(j)}$ be the number of j-particles which have entered the system between times 0 and t, and define

$$T_t^{(j)} := \int_0^t \eta_s^{(j)}(2)ds$$

and

$$\tilde{T}_t := \int_0^t \eta_s^{(1)}(2)(1-\eta_s^{(1)}(1))ds,$$

for all $j \in \{1, \dots, i\}$.

$T_t^{(j)}$ is the time spent by j-particles in site 2 during $[0,t]$, and $\tilde{T}_t$ is the length of the subset of $[0,t]$ for which 2-particles can enter site 1 with rate ϵ (excepted if the site 1 is already occupied by another 2-particle). The following lemma says that we have a uniform control on the total time spent by a particular particle of type ≥ 2 at site 2. Let $T_\infty^{(j),k}$ be the total time spent in site 2 by the k-th particle of type $j \ge 2$ which have entered the system.

Lemma 4.1 *There exists a constant $C_\lambda \in]0,+\infty[$, independent of ϵ, such that for all $k \ge 1$ and all $j \ge 2$ we have*

$$\mathbf{E}\left[T_\infty^{(j),k}\right] \le C_\lambda.$$

Proof Let E_t be the event that, between times t and $t+1$, a first class particle enters (or tries to enter) the system, then jumps, if it is possible, to site 2, and finally another first class particle tries to enter the system. We also assume that in E_t there is no other jump time for $\mathcal{N}_1, \mathcal{N}_2$ and $\mathcal{N}_1^b$ between 0 and t. In particular, if E_t occurs and if there was a particle of type greater or equal to 2 in site 2 at time t, then it has disappeared at time $t+1$. $q(\lambda) := \mathbf{P}[E_t]$ does not depend on t neither on ϵ and $q(\lambda) > 0$.

On the event $\{T_\infty^{(j),k} > t\}$, there exists a time τ such that the k-th particle of type j is at the site 2 and it has spent exactly time t in this site between 0 and τ. We have $E_\tau \subset \{T_\infty^{(j),k} \le t+1\}$. Hence

$$\mathbf{P}\left[E_\tau | T_\infty^{(j),k} > t\right] \le \mathbf{P}\left[T_\infty^{(j),k} \le t+1 | T_\infty^{(j),k} > t\right]. \tag{7}$$

But τ is a stopping time for the Markov process $(\eta_t^{(l)}, l = 1, \dots, j)_{t\ge 0}$ and the event E_τ depends only on the Poisson processes of the Harris system for times between τ and $\tau+1$, so, conditionally to $\{\tau < \infty\}$, E_τ has the same law as E_0 by the strong Markov property. Hence the left-hand side of (7) is equal to $q(\lambda)$. Finally, we have

$$\mathbf{P}\left[T_\infty^{(j),k} > t+1\right] \le (1-q(\lambda))\mathbf{P}\left[T_\infty^{(j),k} > t\right].$$

The last inequality implies that there exist some deterministic positive constants a_1, a_2, depending only on λ, such that almost surely and for all $t \geq 0$ we have

$$\mathbf{P}\left[T_\infty^{(j),k} > t\right] \leq a_1 e^{-a_2 t}.$$

The result follows with $C_\lambda := \int_0^\infty a_1 e^{-a_2 t} dt$. □

Finally, the following theorem gives the estimates that we need:

Theorem 4.2 *For each $1 \leq j \leq i$ and $k \geq i$, $T_t^{(j)}/t$ converges almost surely to a deterministic value if the process starts under $\mu_\infty^{(k)}$. Furthermore, for all $\epsilon < \frac{1}{2C_\lambda}$, where C_λ is as in Lemma 4.1, we have*

$$\limsup_{t\to\infty} \frac{N_t^{(j)}}{t} \leq c_{j-1}\epsilon^{j-1}, \qquad \lim_{t\to\infty} \frac{T_t^{(j)}}{t} \leq c_j \epsilon^{j-1},$$

for $1 \leq j \leq i-1$, and

$$\limsup_{t\to\infty} \frac{N_t^{(i)}}{t} \leq 2c_{i-1}\epsilon^{i-1}, \qquad \lim_{t\to\infty} \frac{T_t^{(i)}}{t} \leq 2c_i \epsilon^{i-1},$$

where $(c_j)_{j=1,\ldots,i}$ are constants (depending only on λ) such that $c_0 := \lambda(1-\lambda)$ and $c_j := C_\lambda^{j-1} c_0$.

Proof We have seen that every $\mu_\infty^{(k)}$ is stationary and ergodic, so by the ergodic theorem, we have almost surely

$$\frac{T_t^{(j)}}{t} \underset{t\to\infty}{\longrightarrow} \mu_\infty^{(k)} \{\eta \in \mathcal{X}_k : \eta(2) = j\} \tag{8}$$

and

$$\frac{\tilde{T}_t}{t} \underset{t\to\infty}{\longrightarrow} \mu_\infty^{(k)} \{\eta \in \mathcal{X}_k : \eta(1) \geq 2, \eta(2) = 1\}. \tag{9}$$

Since the distribution of the first class particles is ν^λ under every $\mu_\infty^{(k)}$, the right-hand side of (8) is λ if $j = 1$ and the right-hand side of (9) is $\lambda(1-\lambda)$. Using Proposition 3.4, $N_t^{(1)}/t$ converges to $\lambda(1-\lambda)$ almost surely.

Let

$$M_t^{(2)} := \sharp\left\{s \in \mathcal{N}_2^b \cap [0,t] : \eta_s^{(1)}(2)\left(1 - \eta_s^{(1)}(1)\right) = 1\right\}.$$

Then almost surely $N_t^{(2)} \leq M_t^{(2)}$ and applying Proposition 3.4:

$$\limsup_{t\to\infty} \frac{N_t^{(2)}}{t} \leq \epsilon\lambda(1-\lambda) = \lim_{t\to\infty} \frac{M_t^{(2)}}{t}. \tag{10}$$

Now, we need to find an upper bound for $\lim_{t\to\infty} T_t^{(2)}/t$. First, we can remark that $T_t^{(2)}$ can be decomposed into two parts: the time spent by initial second class particles, *i.e.*, particles present at time 0, denoted by $T_{t,1}^{(2)}$, plus the time spent by the new second class particles in site 2, denoted by $T_{t,2}^{(2)}$. But, since $T_{t,1}^{(2)}$ is bounded by a random variable that is almost surely finite, it is sufficient to study $\lim_{t\to\infty} T_{t,2}^{(2)}/t$. Indeed, using $\lambda + \epsilon < 1/2$, it can be shown that every initial second class particle has a probability uniformly bounded from below by a positive constant, to never go behind its starting point (see [7]). Thus the number of initial second class particles visiting the site 2 is finite and each of them spent a finite time in this site as a consequence of Lemma 4.1.

As we have seen previously, $\mu_\infty^{(k)} \prec \nu^{\lambda+\epsilon}$. The idea is that since we know the number of second class particles created up to time t, it is sufficient to bound the time spent in site 2 by one of them in the environment $\nu^{\lambda+\epsilon}$ where it is slower. But there are some difficulties. For example, at the moment where a second class particle is created, the environment in $\{2,3,\dots\}$ is not dominated anymore by a Bernoulli product measure with density $\lambda+\epsilon$ because we know that a first class particle has to be in site 2. To avoid this problem, we will use the following fact: if a particle of a class different than 1 is at site 2 at time t then it has a positive probability (depending only on λ) to be out of the system at time $t+1$. This implies Lemma 4.1 which says:

$$\mathbf{E}\left[T_\infty^{(2),l}\right] \le C_\lambda, \tag{11}$$

where C_λ is a constant. Take any $\beta > \epsilon\lambda(1-\lambda)$ and

$$\tau := \inf\left\{t \ge 0 : \forall s \ge t,\, N_s^{(2)} \le \beta s\right\}.$$

We have that τ is almost surely finite by (10) and

$$\frac{T_{t,2}^{(2)}}{t}\mathbf{1}_{\{\tau\le t\}} \le \frac{1}{t}\sum_{k=1}^{N_t^{(2)}} T_\infty^{(2),k}\mathbf{1}_{\{\tau\le t\}} \le \frac{1}{t}\sum_{k=1}^{\lfloor\beta t\rfloor} T_\infty^{(2),k}\mathbf{1}_{\{\tau\le t\}}. \tag{12}$$

Taking expectation in both sides, it leads to

$$\mathbf{E}\left[\frac{T_{t,2}^{(2)}}{t}\mathbf{1}_{\{\tau\le t\}}\right] \le \frac{1}{t}\sum_{k=1}^{\lfloor\beta t\rfloor}\mathbf{E}\left[T_\infty^{(2),k}\mathbf{1}_{\{\tau\le t\}}\right] \underset{(11)}{\le} \frac{\lfloor\beta t\rfloor}{t}C_\lambda. \tag{13}$$

Hence, by dominated convergence we have almost surely

$$\lim_{t\to\infty}\frac{T_t^{(2)}}{t} = \lim_{t\to\infty}\frac{T_{t,2}^{(2)}}{t} = \lim_{t\to\infty}\mathbf{E}\left[\frac{T_{t,2}^{(2)}}{t}\mathbf{1}_{\{\tau\le t\}}\right] \le \beta C_\lambda. \tag{14}$$

The above inequality is true for all $\beta > \epsilon\lambda(1-\lambda)$, thus we also have

$$\lim_{t\to\infty}\frac{T_t^{(2)}}{t} \le \epsilon\lambda(1-\lambda)C_\lambda.$$

Let now $c_2 := C_\lambda c_1$ and by induction, using exactly the same arguments, we have for all $1\le j\le i-1$:

$$\limsup_{t\to\infty}\frac{N_t^{(j)}}{t} \le c_{j-1}\epsilon^{j-1},$$

and

$$\lim_{t\to\infty}\frac{T_t^{(j)}}{t} \le c_j\epsilon^{j-1},$$

where $c_j := C_\lambda^{j-1}\lambda(1-\lambda)$.

Finally, let $\alpha := \limsup_{t\to\infty} N_t^{(i)}/t$. Doing the same computation as in (12), (13) and (14), we get:

$$\lim_{t\to\infty} \frac{T_t^{(i)}}{t} \leq \alpha C_\lambda.$$

Consequently,

$$\lim_{t\to\infty} \frac{T_t^{(i-1)} + T_t^{(i)}}{t} \leq c_{i-1}\epsilon^{i-2} + \alpha C_\lambda,$$

which implies as in (10):

$$\limsup_{t\to\infty} \frac{N_t^{(i)}}{t} = \alpha \leq (c_{i-1}\epsilon^{i-2} + \alpha C_\lambda)\epsilon.$$

Since $\epsilon < \frac{1}{2C_\lambda}$, we have $\alpha \leq 2c_{i-1}\epsilon^{i-1}$ and

$$\lim_{t\to\infty} \frac{T_t^{(i)}}{t} \leq 2c_i\epsilon^{i-1}.$$

□

Now, let $\bar{N}_t^{(i-1)}$ and $\bar{N}_t^{(i)}$ be the number of particles which have entered the system between 0 and t for the processes $(\bar{\eta}_t^{(i-1)})_{t\geq 0}$ and $(\bar{\eta}_t^{(i)})_{t\geq 0}$. We deduce from the above theorem that

$$\limsup_{t\to\infty} \frac{\bar{N}_t^{(i)} - \bar{N}_t^{(i-1)}}{t} = \limsup_{t\to\infty} \frac{N_t^{(i)}}{t} = O(\epsilon^{i-1}).$$

4.2 The Asymptotic Flux at the First Order

In this section, we consider the particle system with generator given by (6) for $i = 3$ (see Fig. 5). In order to differentiate it from particle systems we will define below, we will now refer to this system as the *true* process. In the previous section we have seen that in order to compute $\lim_{t\to\infty} \bar{N}_t^{(i)}/t$ up to order ϵ, it is sufficient to compute this limit only for first and second class particles. In other words, if $\mathbf{N}_t^{(j)}$ denotes the number of new j-particles, *i.e.*, the number of j-particles at time t which was not in the system at time 0, then:

$$\begin{aligned}\limsup_{t\to\infty} \frac{\mathbf{N}_t^{(1)} + \mathbf{N}_t^{(2)} + \mathbf{N}_t^{(3)}}{t} &= \limsup_{t\to\infty} \frac{\mathbf{N}_t^{(1)} + \mathbf{N}_t^{(2)}}{t} + o(\epsilon),\\ &= \lambda(1-\lambda) + \limsup_{t\to\infty} \frac{\mathbf{N}_t^{(2)}}{t} + o(\epsilon).\end{aligned}$$

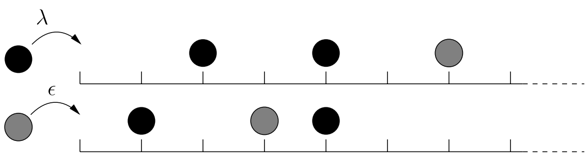

Fig. 5 First class particles, in black, enter with rate λ whatever is the configuration in {2, 3, ...} and second class particles, in grey, enter with rate ϵ if the site 2 is occupied by a first class particle

In the following we denote by N_t, rather than by $\mathbf{N}_t^{(2)}$, the number of new second class particles because there will be no possible confusion. The aim of this section is to prove a law of large numbers for N_t and to compute the limit up to order ϵ. First we introduce some notation. Let $c > 0$ such that $\lambda + c < \frac{1}{2}$ and $\epsilon \in [0, c]$. Consider the point process $\mathcal{N}_2^b \cap \{t \geq 0 : \eta_t(1) \neq 1, \eta_t(2) = 1\}$, and denote its elements ordered chronologically by $\tau_1^e < \cdots < \tau_i^e < \cdots$. By construction, at each time τ_i^e, a second class particle tries to enter the system. We denote by $X_i(t)$ the position at time t of this particle, with the convention $X_i(t) := 0$ if the corresponding particle is not in the system at time t. We define

$$\tau_i^s := \inf\{t \geq \tau_i^e : X_i(t) = 0\}, \qquad S_i(t) := \mathbf{1}_{X_i(t) \geq 1}, \quad \text{and} \quad S_i := \mathbf{1}_{\tau_i^s = \infty}.$$

Remark that there is a positive probability that $\tau_i^s = \tau_i^e$. This happens if $\eta_{\tau_i^e}(1) = 2$. In this case, $X_i(t) = 0$ for all $t \geq 0$.

In order to have simpler estimates in the sequel, we consider the process $(\eta_t)_{t \geq 0}$ on $\mathcal{X}_3$ starting with the measure $\mu_\infty^{(3)}(.|\eta(1) \neq 1, \eta(2) = 1)$. Of course, the limit that we obtain in this case is the same as the one we would get if we started from $\mu_\infty^{(3)}$. Moreover, the estimates of Theorem 4.2 also hold in this case. Indeed, the distribution of the process converges to $\mu_\infty^{(3)}$. In the sequel, we denote $\bar{\eta}_t(x) := \mathbf{1}_{\eta_t(x) \neq \infty}$ the process with indistinguishable particles associated to $(\eta_t)_{t \geq 0}$.

Since $\nu^\lambda \prec \mu_\infty \prec \nu^{\lambda + c}$ and the dynamic is monotone, we can make a basic coupling with a TASEP(λ), denoted $\eta_\cdot^{inf}$, and a TASEP($\lambda + c$), denoted $\eta_\cdot^{sup}$, such that:

- η_0^{inf} has distribution $\nu^\lambda(.|\eta(1) = 0, \eta(2) = 1)$,
- η_0^{sup} has distribution $\nu^{\lambda+c}(.|\eta(1) = 0, \eta(2) = 1)$,
- almost surely $\eta_t^{inf} \leq \bar{\eta}_t \leq \eta_t^{sup}$, for all $t \geq 0$.

4.2.1 The Process Without Interaction

We define a new particle system with state space $\{0, 1, (2, i)_{i \geq 1}\}^{\mathbb{Z}_+^*}$ and the following generator:

$$\begin{aligned}\bar{\Omega}_\nu f(\eta) := {} & \nu \mathbf{1}_{\eta(1) \neq 1}(f(\eta_{1\to 1}) - f(\eta)) + \epsilon \mathbf{1}_{\eta(1) \neq 1, \eta(2) = 1}(f(\eta_{2 \to 1}) - f(\eta)) \\ & + \sum_{x=1}^{\infty} \mathbf{1}_{\eta(x) \neq 0, \eta(x+1) \neq 1}(f(\eta^{x,x+1}) - f(\eta)), \end{aligned} \tag{15}$$

for all cylindrical function f, where

$$\eta_{1 \to 1}(z) := \begin{cases} 1 & \text{if } z = 1, \\ \eta(z) & \text{otherwise}, \end{cases} \tag{16}$$

$$\eta_{2 \to 1}(z) := \begin{cases} (2, 1) & \text{if } z = 1 \text{ and } \eta(1) = 0, \\ (2, i+1) & \text{if } z = 1 \text{ and } \eta(1) = (2, i), \\ \eta(z) & \text{otherwise}, \end{cases} \tag{17}$$

and

$$\eta^{x,x+1}(z) := \begin{cases} \eta_{x,x+1}(z) & \text{if } \eta(x) = 1, \\ 0 & \text{if } \eta(x) \neq 1 \text{ and } z = x, \\ (2, i+j) & \text{if } z = x+1, \eta(x) = (2, i) \text{ and } \eta(x+1) = (2, j), \\ \eta(z) & \text{otherwise}, \end{cases}$$

with the convention $(2, 0) := 0$ (j can be equal to 0). We will refer to this process as the process *without interaction*.

This particle system has the following description: there are two classes of particles; first class particles perform a TASEP(ν); second class particles enter with rate ϵ if a first class particle is in site 2 and with rate 0 otherwise; they have lower priority than first class particles; and, contrary to the process of Sect. 4.1, second class particles are allowed to jump on a site containing one or more second class particles. Once some particles (necessarily of type 2) are on the same site at a given time, they will always jump together since they use the same Harris system. Another possible choice would be to put a Poisson clock on particles instead of sites. This would lead to the same asymptotic results.

To link our process to the above one, we proceed as follows. We construct a process $\xi^{inf}_{\cdot}$ on $\{0, 1, (2, i)_{i\geq 1}\}^{\mathbb{Z}_+^*}$, with generator $\bar{\Omega}_\lambda$, in such a way that the process $\eta^{inf}_{\cdot}$ defined above is exactly the process of first class particles of $\xi^{inf}_{\cdot}$. Furthermore, at each time τ_i^e, we add a second class particle in $\xi^{inf}_{\cdot}$ at site 1 and we denote by $X_i^{inf}(t)$ its trajectory. This particle will behave as a second class particle in the system with generator (15), *i.e.*, it has a lower priority than first class particles but it can jump on a site already occupied by an other second class particle. As a consequence, we can remark that, contrary to $X_i(\tau_i^e)$, we have almost surely $X_i^{inf}(\tau_i^e) \geq 1$. By construction we almost surely have $X_i(t) \leq X_i^{inf}(t)$ for all $t \geq 0$. Indeed, ξ^{inf} and η have the same first class particles and contrary to X_i^{inf}, the particle X_i is blocked by other second class particles thus it stays behind X_i^{inf}. In order to bound from below the trajectory $X_i(t)$, we now construct a process ξ^{sup} on $\{0, 1, 2\}^{\mathbb{Z}_+^*}$ such that for all $t \geq 0, x \geq 1$, $\mathbf{1}_{\xi_t^{sup}(x)\neq 0} = \eta_t^{sup}(x)$, by affecting the type 2 to particles of $\eta^{sup}_{\cdot}$ entering at times $(\tau_i^e)_{i\geq 1}$. In the same way, we denote by $X_i^{sup}(t)$ their trajectory and we have almost surely for all $t \geq 0$, $X_i^{sup}(t) \leq X_i(t)$. We define analogously the quantities N_t^{inf}, N_t^{sup}, $\tau_i^{s,inf}$, $\tau_i^{s,sup}$, etc.

Consider the following initial configuration: at time 0, first class particles are distributed on $\mathbb{Z}_+^* \backslash \{1, 2\}$ according to ν^λ (the Bernoulli product measure with density λ), and we put one first class particle in site 2 and one second class particle in site 1. We show in Proposition 4.6 below that this is exactly the distribution of the configuration $\eta^{inf}_{\tau_i^e}$ for all $i \geq 1$. Then first class particles enter site 1 with rate λ and they have priority over the second class particle. Two cases can occur: either the second class particle survives, or it dies. Let $p(\lambda)$ be the probability that the second class particle survives. p is a non-increasing function, $p(0) = 1$, $p(\frac{1}{2}) = 0$ and $p(\lambda) > 0$ for all $\lambda < \frac{1}{2}$. Indeed, for the last point, it can be shown that if the second class particle survives, then it has a positive speed $1 - 2\lambda$ (see e.g. [7]). The exact expression of $p(\lambda)$ is unknown. However, simulations indicate that $p(\lambda) = 1 - 2\lambda$ for $\lambda \in [0, \frac{1}{2}]$. We have by construction and with results of Sect. 4.2.2 below, $\mathbf{P}[S_i^{inf} = 1] = p(\lambda)$ and $\mathbf{P}[S_i^{sup} = 1] = p(\lambda + c)$ for all $i \geq 1$. Consequently, $p(\lambda + c) \leq \mathbf{P}[S_i = 1] \leq p(\lambda)$.

The aim of Sect. 4.2 is to prove the following law of large numbers:

Theorem 4.3 *Almost surely*, $\lim_{\epsilon\downarrow 0} \frac{1}{\epsilon} \lim_{t\to\infty} \frac{N_t}{t} = \lambda(1 - \lambda)p(\lambda)$.

With the discussion at the beginning of Sect. 4.2, Theorem 1.2 follows.

The idea is the following: when ϵ is very small, second class particles do not interact before they are very far from the left boundary and if a second class particle is far enough from this boundary, then it survives with high probability. In other words, the effect on N_t of interaction goes to 0 with ϵ. The first step in the proof will be to find estimates for the process without interaction and to prove the theorem in this case. Next, we will show, for the true process, that if two second class particles meet, they both survive with a probability going to 1 as ϵ goes to 0; this implies the theorem.

4.2.2 *Distribution of the Process at Time τ_i^e*

In this section we prove that at each time τ_i^e, the process $\eta^{inf}_{\tau_i^e}$ has distribution $\nu^\lambda(.|\eta_0(2)(1-\eta_0(1))=1)$. For that we need some preliminary results about the motion of a tagged particle in a TASEP. It is convenient to regard the exclusion process as a Markov process (X_t,η_t) on the space $V:=\{(x,\eta)\in\mathbb{Z}_+^*\times X:\eta(x)=1\}$, so that x is the position of the tagged particle and η is the entire configuration. Consider the generator

$$\Omega f(x,\eta):=\sum_{y\in\mathbb{Z}_+^*,y\neq x}\eta(y)(1-\eta(y+1))\left[f(x,\eta_{y.y+1})-f(x,\eta)\right]$$
$$+(1-\eta(x+1))\left[f(x+1,\eta_{x.x+1})-f(x,\eta)\right], \tag{18}$$

for all cylindrical functions. Suppose that initially, the tagged particle is placed at some point $x\in\mathbb{Z}_+^*$ and other particles are placed according to the Bernoulli product measure with density λ on $\mathbb{Z}_+^*\backslash\{x\}$. Then the system is stationary when viewed from the position of the tagged particle. In other words, for all $t\geq 0$,

$$\mathbf{E}\left[\prod_{y\in A}\eta_t(\phi_t(y))\right]=\lambda^{|A|},$$

where A is a finite subset of $\mathbb{Z}_+^*$ and

$$\phi_t(y):=\begin{cases}y & \text{if } y<X_t,\\ y+1 & \text{if } y\geq X_t.\end{cases}$$

Moreover, the random variable $\prod_{y\in A}\eta_t(\phi_t(y))$ is independent of X_t for each t. Consequently, it can be shown that X_t-X_0 is a Poisson process with parameter $1-\lambda$ (see [5]).

The following proposition will be useful to describe the process at a random time.

Proposition 4.4 *Let X_t be the position of a tagged particle starting at site* 0. *The other particles are initially distributed according to a Bernoulli product measure with density λ on* $\{1,2,\ldots\}$. *Let* $H_0:=0$, *and for $i\geq 1$, let* $H_i:=\inf\{t\geq 0:X_t=i\}$. *Then for all* $i\geq 0$, $(\eta_{H_i}(X_{H_i}+x))_{x\geq 0}$ *has the same distribution as* η_0.

Proof By the strong Markov property, it is sufficient to prove it for $i=1$ since it is true for $i=0$ by hypothesis. Define $X_t^0:=X_t$ and for $i\geq 1$, X_t^i is the position of the i-th particle to the right of X_t (X_t^i always exists if $\lambda>0$ and if not the result is obvious). The result will follow if we can prove that $X^1_{H_1}-X^0_{H_1},\ldots,X^L_{H_1}-X^{L-1}_{H_1}$ are i.i.d. random variables with geometric distribution with parameter λ for all $L\geq 1$. Since X_t-X_0 is a Poisson process with parameter $1-\lambda$, the process $\xi_t(i):=X_t^{i+1}-X_t^i-1$, for $i=0,\ldots,L-1$, is a totally asymmetric Zero Range process on $\{0,\ldots,L-1\}$ with generator

$$\Omega f(\xi):=\sum_{y=0}^{L-1}\mathbf{1}_{\xi(y)\geq 1}\left[f(\xi^y)-f(\xi)\right]+(1-\lambda)\left[f(\xi^L)-f(\xi)\right], \tag{19}$$

where

$$\xi^y(z):=\begin{cases}\xi(z) & \text{if } z\notin\{y-1,y\},\\ \xi(y)-1 & \text{if } z=y,\\ \xi(y-1)+1 & \text{if } z=y-1.\end{cases}$$

Let μ be the product measure on $\mathbb{N}^L$ such that $\mu\{\xi : \xi(0) = k\} = \lambda(1-\lambda)^k$. μ is invariant for $\xi_.$ and $\xi_0 \sim \mu$. We also have

$$H_1 = \inf\{t \geq 0 : \xi_{t^-}(0) = \xi_t(0) + 1\},$$

i.e., H_1 is the first time at which a particle leaves the system (from 0). We need to prove that ξ_{H_1} has distribution μ. Let $q(\xi, \zeta)$ be the rate for which $(\xi_t)_{t\geq 0}$ goes from ξ to ζ, for every $\xi, \zeta \in \mathbb{N}^L$, and let $q(\xi) := \sum_\zeta q(\xi, \zeta)$.

Fix a configuration $\gamma \in \mathbb{N}^L$ and let $\phi(\xi) := \mathbf{P}[\xi_{H_1} = \gamma | \xi_0 = \xi]$. Conditioning on the first step we get:

$$\phi(\xi) = \sum_\zeta \frac{q(\xi,\zeta)}{q(\xi)} \mathbf{1}_{\zeta(0)\geq\xi(0)} \phi(\zeta) + \frac{q(\xi,\gamma)}{q(\xi)} \mathbf{1}_{\gamma=\xi^0}. \tag{20}$$

Moreover, since μ is invariant, $\int \Omega\phi(\xi) d\mu = 0$, thus

$$\begin{aligned}\sum_{\xi,\zeta} \mu(\xi) q(\xi,\zeta)\phi(\zeta) &= \sum_{\xi,\zeta} \mu(\xi) q(\xi,\zeta)\phi(\xi) = \sum_\xi \mu(\xi) q(\xi)\phi(\xi),\\ &\overset{(20)}{=} \sum_{\xi,\zeta} \mu(\xi) q(\xi,\zeta)\mathbf{1}_{\zeta(0)\geq\xi(0)}\phi(\zeta) + \sum_\xi \mu(\xi) q(\xi,\gamma)\mathbf{1}_{\gamma=\xi^0}. \end{aligned} \tag{21}$$

But $q(\xi,\zeta)\mathbf{1}_{\zeta(0)<\xi(0)} = 1$ if $\zeta = \xi^0$ and 0 otherwise, hence

$$\begin{aligned}\sum_\xi \mu(\xi)\phi(\xi^0)\mathbf{1}_{\xi(0)\geq 1} &= \sum_\xi \mu(\xi) q(\xi,\gamma)\mathbf{1}_{\gamma=\xi^0},\\ &= \mu(\xi : \xi^0 = \gamma) = (1-\lambda)\mu(\gamma). \end{aligned} \tag{22}$$

Finally, the left-hand side of (22) is equal to

$$(1-\lambda)\sum_\xi \mu(\xi^0)\phi(\xi^0)\mathbf{1}_{\xi(0)\geq 1} = (1-\lambda)\int \phi(\xi) d\mu,$$

which leads to $\mathbf{P}[\xi_{H_1} = \gamma] = \mu(\gamma)$. □

Corollary 4.5 *Consider the TASEP on $\mathbb{Z}_+^*$ starting from $\nu^\lambda(.|\eta(2)(1-\eta(1)) = 1)$. Let H_i be the time at which the first particle created is at site i, for $i \geq 1$. Then $(\eta_{H_i}(i+x))_{x\geq 1}$ has distribution ν^λ.*

Proof By Proposition 4.4, it is sufficient to treat the case $i = 1$. The distance d between the initial particle at site 2 and the new particle evolves as follow: it increases by 1 with rate $1-\lambda$ and decreases by 1 with rate λ until the new particle is at site 1. Hence, at this time, $d+1$ is distributed as a geometric random variable with parameter λ. Using again Proposition 4.4, the configuration in front of the first particle has for distribution a Bernoulli product measure with parameter λ. Therefore, it is the same for the new particle. □

Now we can give the distribution of $\eta^{inf}_{\tau_i^e}$.

Proposition 4.6 *For each $i \geq 1$, $\eta^{inf}_{\tau_i^e}$ has distribution $\nu^\lambda(.|\eta_0(2)(1-\eta_0(1)) = 1)$. In particular, it does not depend on ϵ.*

Proof We use the following compact notation for initial measures: $\nu^\lambda(.|\eta_0(2)(1-\eta_0(1))=1)$ will be denoted by $0\,1\,\nu^\lambda$, and $\nu^\lambda(.|\eta_0(1))=1)$ by $1\,\nu^\lambda$.

Let f be a bounded function on $\{0,1\}^{\mathbb{Z}_+^*}$. Conditioning on the type of the first new particle and using the above corollary with the Markov property we get:

$$\mathbf{E}^{0\,1\,\nu^\lambda}\left[f(\theta^2\eta_{\tau_i^e})\right]=\frac{\epsilon}{1-\lambda+\epsilon}\mathbf{E}^{2\,1\,\nu^\lambda}\left[f(\theta^2\eta_0)\right]+\frac{1-\lambda}{1-\lambda+\epsilon}\mathbf{E}^{1\,\nu^\lambda}\left[f(\theta^2\eta_{\tau_i^e})\right].$$

The first expectation on the right-hand side is equal to $\langle f\rangle_{\nu^\lambda}$ and, using Proposition 4.4, the second expectation is equal to $\mathbf{E}^{0\,1\,\nu^\lambda}[f(\theta^2\eta_{\tau_i^e})]$. Hence $\mathbf{E}^{0\,1\,\nu^\lambda}[f(\theta^2\eta_{\tau_i^e})]=\langle f\rangle_{\nu^\lambda}$. □

4.2.3 Proof in the Case "Without Interaction"

Consider a family $(\mathcal{N}_\lambda^b)_{0\le\lambda<\frac{1}{2}}$ of Poisson point processes such that the parameter of $\mathcal{N}_\lambda^b$ is λ and for all $0\le\lambda\le\mu<\frac{1}{2}$, $\mathcal{N}_\lambda^b\subset\mathcal{N}_\mu^b$ and $\mathcal{N}_\mu^b\backslash\mathcal{N}_\lambda^b$ is independent of $\mathcal{N}_\lambda^b$. Take also a family $(\eta_0^\lambda)_{0\le\lambda<\frac{1}{2}}$ of initial configurations such that $\eta_0^\lambda(2)(1-\eta_0^\lambda(1))=1$ for all $\lambda\in[0,\frac{1}{2}[$, the distribution of η_0^λ on $\{3,4,\dots\}$ is ν^λ, and for all $x\ge 3$ and all $0\le\lambda\le\mu<\frac{1}{2}$, $\eta_0^\lambda(x)\le\eta_0^\mu(x)$ almost surely. Then using the same Poisson point processes $(\mathcal{N}_x, x\ge 1)$ for the bulk dynamic we construct, as in Sect. 2, the family of TASEP $(\eta_.^\lambda)_{0\le\lambda<\frac{1}{2}}$ such that $\eta_.^\lambda$ is a TASEP(λ) and for all $t\ge 0$ and all $0\le\lambda\le\mu<\frac{1}{2}$, $\eta_t^\lambda\le\eta_t^\mu$ almost surely. At time 0 we add a second class particle in site 1 to each of these processes and we denote by $X_\lambda(t)$ the position at time t of the particle in the process $\eta_.^\lambda$ (with the convention $X_\lambda(t):=0$ if the particle has left the system). We define

$$S^\lambda := \mathbf{1}_{X_\lambda\text{ survives}},$$

and

$$H_x^\lambda := \inf\{t\ge 0 : X_\lambda(t)=x\},$$

for all $x\ge 1$.

Since we use the basic coupling, the following inequality holds almost surely:

$$X_\lambda(t)\ge X_\mu(t),$$

for all $\lambda\le\mu$ and all $t\ge 0$. This easily implies that, for all $\lambda\le\mu$ and all $x\ge 1$, $S^\lambda\ge S^\mu$ and $H_x^\lambda\le H_x^\mu$. Furthermore, by definition of $p(.)$, S^λ is a Bernoulli random variable with parameter $p(\lambda)$.

We start with an intuitive lemma which will be useful to propagate results from the process without interaction to the true process.

Lemma 4.7 *The function* $p:[0,1]\to[0,1]$ *is right-continuous.*

Proof Since $p(\lambda)=0$ for $\lambda\ge\frac{1}{2}$, it is sufficient to prove it on $[0,\frac{1}{2}[$. Let $0\le\lambda<\frac{1}{2}$, $\epsilon'>0$ and $0<c<\frac{1}{2}-\lambda$. There exists some $x\ge 1$ such that

$$\mathbf{P}\left[S^{\lambda+c}=0|H_x^{\lambda+c}<\infty\right]<\epsilon'.$$

Indeed, if $M:=\max\{X_{\lambda+c}(t), t\ge 0\}$ then conditionally to $\{S^{\lambda+c}=0\}$, M is almost surely finite. Thus there exists $x\ge 1$ such that

$$\mathbf{P}\left[M\ge x|S^{\lambda+c}=0\right]<\epsilon'\frac{p(\lambda+c)}{1-p(\lambda+c)}.$$

Then, using

$$\{H_x^{\lambda+c} < \infty\} = \{M \geq x\},$$

and

$$\mathbf{P}[M \geq x] \geq \mathbf{P}\left[S^{\lambda+c} = 1\right] = p(\lambda + c),$$

this implies

$$\mathbf{P}\left[S^{\lambda+c} = 0 | H_x^{\lambda+c} < \infty\right] = \mathbf{P}\left[M \geq x | S^{\lambda+c} = 0\right] \frac{\mathbf{P}[S^{\lambda+c} = 0]}{\mathbf{P}[M \geq x]} < \epsilon'.$$

Furthermore for all $\epsilon \in [0, c]$,

$$\begin{aligned}
\mathbf{P}\left[S^{\lambda+\epsilon} = 0 | H_x^{\lambda+\epsilon} < \infty\right] &= \frac{\mathbf{P}[H_x^{\lambda+\epsilon} < \infty] - \mathbf{P}[S^{\lambda+\epsilon} = 1]}{\mathbf{P}[H_x^{\lambda+\epsilon} < \infty]}, \\
&= \frac{\mathbf{P}[S^{\lambda+\epsilon} = 0] - 1}{\mathbf{P}[H_x^{\lambda+\epsilon} < \infty]} + 1, \\
&\leq \frac{\mathbf{P}[S^{\lambda+c} = 0] - 1}{\mathbf{P}[H_x^{\lambda+c} < \infty]} + 1, \\
&= \mathbf{P}\left[S^{\lambda+c} = 0 | H_x^{\lambda+c} < \infty\right] < \epsilon'. \qquad (23)
\end{aligned}$$

Now let $t_0 \geq 0$ such that

$$\mathbf{P}\Big[\sup_{t \in [0, t_0]} X_\lambda(t) \geq x | H_x^\lambda < \infty\Big] > 1 - \epsilon'. \qquad (24)$$

We can find $0 < c' \leq c$ such that

$$\mathbf{P}\left[\sum_{i=1}^{x} (\eta_0^{\lambda+c'}(i) - \eta_0^\lambda(i)) = 0,\ (\mathcal{N}_{\lambda+c'}^b \backslash \mathcal{N}_\lambda^b) \cap [0, t_0] = \emptyset\right] > 1 - \epsilon'. \qquad (25)$$

We define the events

$$B := \left\{\sum_{i=1}^{x} (\eta_0^{\lambda+c'}(i) - \eta_0^\lambda(i)) = 0,\ (\mathcal{N}_{\lambda+c'}^b \backslash \mathcal{N}_\lambda^b) \cap [0, t_0] = \emptyset\right\},$$

and

$$A := \left\{\sup_{t \in [0, t_0]} X_\lambda(t) \geq x\right\} \cap B.$$

By the Harris construction of the process, the event B is independent of $\{H_x^\lambda < \infty\}$ and $\{\sup_{t \in [0, t_0]} X_\lambda(t) \geq x\}$. Moreover, $A \subset \{H_x^{\lambda+c'} < \infty\}$, thus using (24) and (25)

$$\begin{aligned}
\mathbf{P}\left[H_x^{\lambda+c'} < \infty | H_x^\lambda < \infty\right] &\geq \mathbf{P}\left[A | H_x^\lambda < \infty\right], \\
&= \mathbf{P}\Big[\sup_{t \in [0, t_0]} X_\lambda(t) \geq x | H_x^\lambda < \infty\Big] \mathbf{P}[B], \\
&\geq (1 - \epsilon')^2 > 1 - 2\epsilon'. \qquad (26)
\end{aligned}$$

Finally, with (23) and (26), we get

$$\begin{aligned}
p(\lambda) - p(\lambda + c') &= \mathbf{P}\left[S^{\lambda+c'} = 0, S^{\lambda} = 1\right], \\
&= \mathbf{P}\left[S^{\lambda+c'} = 0, S^{\lambda} = 1, H_x^{\lambda+c'} < \infty\right] + \mathbf{P}\left[S^{\lambda} = 1, H_x^{\lambda+c'} = \infty\right], \\
&\leq \mathbf{P}\left[S^{\lambda+c'} = 0, H_x^{\lambda+c'} < \infty\right] + \mathbf{P}\left[H_x^{\lambda} < \infty, H_x^{\lambda+c'} = \infty\right], \\
&\leq \mathbf{P}\left[S^{\lambda+c'} = 0 | H_x^{\lambda+c'} < \infty\right] + \mathbf{P}\left[H_x^{\lambda+c'} = \infty | H_x^{\lambda} < \infty\right], \\
&< 3\epsilon'.
\end{aligned}$$

□

Now we prove Theorem 4.3 in the case without interaction.

Proposition 4.8 *$\frac{N_t}{t}$ and $\frac{N_t^{inf}}{t}$ both have almost sure limits as t goes to infinity and*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \lim_{t \to \infty} \frac{N_t^{inf}}{t} = \lambda(1-\lambda)p(\lambda).$$

Proof We use the coupling of $\eta_{\cdot}$ with the processes $\xi_{\cdot}^{inf}$ and $\xi_{\cdot}^{sup}$ defined in Sect. 4.2.3.

Recall that N_t is the number of 2-particles at time t which are not in the system at time 0 for the *true* process $(\eta_t)_{t\geq 0}$, *i.e.*, the number of X_i for which $\tau_i^e \leq t < \tau_i^s$. N_t^{inf} and N_t^{sup} have the same definition as N_t but for processes $\xi_{\cdot}^{inf}$ and $\xi_{\cdot}^{sup}$ respectively, or for particles X_i^{inf} and X_i^{sup} respectively.

Let $\bar{N}_t := \sharp(\mathcal{N}_2^b \cap \{t \geq 0 : \eta_t(1) \neq 1, \eta_t(2) = 1\})$. $\bar{N}_t$ is the number of 2-particles (in the process $\eta_{\cdot}$) which have entered the system between time 0 and time t, *i.e.*, the number of X_i for which $\tau_i^e \leq t$.

The convergence to almost sure limits is a consequence of Proposition 3.4. Indeed, for example N_t counts the number of elements of $\mathcal{N}_2^b$ for which $\eta_t(1) \neq 1$ and $\eta_t(2) = 1$ minus the number of elements of $\mathcal{N}_1^b$ for which $\eta_t(1) = 2$. Furthermore, by Proposition 3.4, $\bar{N}_t/t$ converges almost surely to $\lambda(1-\lambda)\epsilon$.

We denote by $(t_n)_{n\geq 1}$ the successive times at which the $\bar{N}_{t_n}$-th 2-particle of the *true* process is exactly the n-th particle which will survive. Then:

$$\frac{N_{t_n}^{inf}}{t_n} = \frac{1}{t_n} \sum_{i=1}^{\bar{N}_{t_n}} S_i^{inf} = \frac{n}{t_n}.$$

Thus if $n_t := \sup\{n \geq 1 : t_n \leq t\}$ then, since t_n/n converges almost surely to $(\lim_{t\to\infty} N_t^{inf}/t)^{-1}$, we almost surely have:

$$\lim_{t \to \infty} \frac{n_t - N_t^{inf}}{t} = 0. \tag{27}$$

On the other hand, n_t is exactly the number of 2-particles which are in the system at time t and which will survive, *i.e.*, $n_t = \sum_{i=1}^{\bar{N}_t} S_i^{inf}$ almost surely. Hence:

$$\lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{\bar{N}_t} S_i^{inf} = \lim_{t \to \infty} \frac{N_t^{inf}}{t}, \quad \text{a.s.}$$

Next we compute the limit of $1/t \sum_{i=1}^{\bar{N}_t} S_i^{inf}$ in expectation. Let $\epsilon' > 0$ and

$$\tau := \inf\left\{t \geq 0 : \forall s \geq t, \left|\frac{\bar{N}_t}{t} - \lambda(1-\lambda)\epsilon\right| < \epsilon'\right\}.$$

τ is almost surely finite and, using $\mathbf{E}[S_i^{inf}] = p(\lambda)$,

$$\begin{aligned}\mathbf{E}\left[\frac{1}{t}\sum_{i=1}^{\bar{N}_t} S_i^{inf}\right] &\geq \frac{1}{t}\sum_{i=1}^{\lfloor(\lambda(1-\lambda)\epsilon-\epsilon')t\rfloor} \mathbf{E}\left[S_i^{inf}\mathbf{1}_{\tau\leq t}\right],\\ &\geq \frac{\lfloor(\lambda(1-\lambda)\epsilon-\epsilon')t\rfloor}{t}p(\lambda) - \frac{1}{t}\sum_{i=1}^{\lfloor(\lambda(1-\lambda)\epsilon-\epsilon')t\rfloor} \mathbf{E}\left[S_i^{inf}\mathbf{1}_{\tau> t}\right].\end{aligned}$$

Since S_i^{inf} is bounded by 1

$$\mathbf{E}\left[\frac{1}{t}\sum_{i=1}^{\bar{N}_t} S_i^{inf}\right] \geq \frac{\lfloor(\lambda(1-\lambda)\epsilon-\epsilon')t\rfloor}{t}(p(\lambda) - \mathbf{P}[\tau > t]).$$

Let t go to infinity, then ϵ' go to 0:

$$\lim_{t\to\infty}\frac{1}{t}\sum_{i=1}^{\bar{N}_t} S_i^{inf} \geq \lambda(1-\lambda)p(\lambda)\epsilon.$$

On the other hand

$$\begin{aligned}\mathbf{E}\left[\frac{1}{t}\sum_{i=1}^{\bar{N}_t} S_i^{inf}\mathbf{1}_{\tau\leq t}\right] &\leq \frac{1}{t}\sum_{i=1}^{\lfloor(\lambda(1-\lambda)\epsilon+\epsilon')t\rfloor} \mathbf{E}\left[S_i^{inf}\mathbf{1}_{\tau\leq t}\right],\\ &\leq \frac{\lfloor(\lambda(1-\lambda)\epsilon+\epsilon')t\rfloor}{t}p(\lambda),\end{aligned}$$

which gives the reverse inequality letting t go to infinity and ϵ' go to 0. Finally

$$\lim_{t\to\infty}\frac{1}{t}\sum_{i=1}^{\bar{N}_t} S_i^{inf} = \lim_{t\to\infty}\frac{N_t^{inf}}{t} = \lambda(1-\lambda)p(\lambda)\epsilon.$$

□

4.2.4 Interaction Implies Survival

The following lemma states that if a second class particle goes far enough, then it survives with high probability.

Lemma 4.9 *For all $\epsilon' > 0$, there exists x_0 (depending only on λ and ϵ') such that if c is small enough, then for all $i \geq 1$*

$$\mathbf{P}\left[\tau_i^s < \infty, \exists t \geq 0, X_i(t) \geq x_0\right] < \epsilon'.$$

Proof We start by proving the same result for X^{inf}. Let

$$M := \sup\left\{X_i^{inf}(t), t \geq 0\right\}.$$

Conditionally on $\{\tau_i^{s,inf} < \infty\}$, M is almost surely finite, thus we can choose x_0 such that

$$\mathbf{P}\left[M \geq x_0 | \tau_i^{s,inf} < \infty\right] < \frac{\epsilon'}{2}.$$

Hence

$$\mathbf{P}\left[\tau_i^{s,inf} < \infty, \exists t \geq 0, X_i^{inf}(t) \geq x_0\right] = \mathbf{P}\left[M \geq x_0 | \tau_i^{s,inf} < \infty\right]\mathbf{P}\left[\tau_i^{s,inf} < \infty\right], < \frac{\epsilon'}{2}.$$

Furthermore, since the law of X_i^{inf} is the same for all $i \geq 1$ (because they enter in the same environment), we can choose the same x_0 for all $i \geq 1$. Then we have, using Lemma 4.7:

$$\begin{aligned}\mathbf{P}\left[\tau_i^s < \infty, \exists t \geq 0, X_i(t) \geq x_0\right] &\leq \mathbf{P}\left[\tau_i^{s,inf} < \infty, \exists t \geq 0, X_i^{inf}(t) \geq x_0\right] + \mathbf{P}\left[S_i \neq S_i^{inf}\right], \\ &< \frac{\epsilon'}{2} + p(\lambda) - p(\lambda + c), \\ &< \epsilon',\end{aligned}$$

if c is small enough. □

For $x \geq 1$, let $H_x := \inf\{t \geq 0 : X_i(t) = x\}$ (we omit the dependence on i in the notation because there will be no possible confusion). We can deduce from this lemma a stronger form of the same estimate:

Corollary 4.10 *Let $x \geq 1$. For all $\epsilon' > 0$, there exists x_1 depending only on λ, ϵ' and x such that if c is small enough, then for all $i \geq 1$*

$$\mathbf{P}\left[H_{x_1} < \infty, \exists t \geq H_{x_1}, X_i(t) \leq x\right] < \epsilon'.$$

Proof We will use the same method as in Lemma 4.1. Let E_t be the following event on the Poisson point processes of the Harris system during the time space $[t, t+1]$:

- one first class particle enters site 1 and moves to site x;
- then one first class particle enters and moves to site $x - 1$;
- we continue in the same way until x first class particles have entered the system and they have moved until that the box $\{1, \ldots, x\}$ is full;
- finally we impose that $\mathcal{N}_x \cap [t, t+1] = \emptyset$.

Then $q_x(\lambda) := \mathbf{P}[E_t]$ depends only on λ and x, is positive and, under this event, every second class particle which was in the box $\{1, \ldots, x\}$ at time t has left the system at time $t + 1$.

Now let x_1 be given by Lemma 4.9 such that

$$\mathbf{P}\left[\tau_i^s < \infty, \exists t \geq 0, X_i(t) \geq x_1\right] < \epsilon' q_x(\lambda),$$

and define $H_x^+ := \inf\{t \geq H_{x_1} : X_i(t) = x\}$. Then

$$\mathbf{P}\left[\tau_i^s < \infty | H_x^+ < \infty\right] \geq q_x(\lambda).$$

This implies

$$\mathbf{P}\left[H_{x_1} < \infty, \exists t \geq H_{x_1}, X_i(t) \leq x\right] = \mathbf{P}\left[H_x^+ < \infty\right] = \frac{\mathbf{P}[\tau_i^s < \infty, H_x^+ < \infty]}{\mathbf{P}[\tau_i^s < \infty | H_x^+ < \infty]},$$
$$< \epsilon'.$$

□

The next lemma states that if we fix $x \geq 1$, then the probability that two second class particles meet in the box $\{1, \ldots, x\}$ goes to 0 with ϵ.

Lemma 4.11 *Let $\tau_{i+1\to i}$ be the first time at which the $(i+1)$-th second class particle tries to jump on the site occupied by the i-th second class particle. Then for all fixed $x \geq 1$,*

$$\mathbf{P}\left[\tau_{i+1\to i} < \infty, X_i(\tau_{i+1\to i}) \leq x\right] \underset{\epsilon\to 0}{\longrightarrow} 0, \quad \textit{uniformly in } i.$$

Proof Fix $\epsilon' > 0$ and let x_1 and $0 < c_0 < \frac{1}{2} - \lambda$ be given by Corollary 4.10 such that

$$\mathbf{P}[H_{x_1} < \infty, \exists t \geq H_{x_1}, X_i(t) \leq x] < \epsilon', \quad \text{for all } \epsilon \leq c_0. \tag{28}$$

Then x_1 and c_0 depend only on λ and ϵ' (and x). We have:

$$\begin{aligned}\mathbf{P}[\exists s \geq t, X_i(s) \in \{1, \ldots, x\}] &\leq \mathbf{P}[\exists s \geq t, X_i(s) \in \{1, \ldots, x\},\ H_{x_1} \leq t] \\ &\quad + \mathbf{P}[X_i(t) \geq 1, H_{x_1} > t], \\ &\leq \mathbf{P}[H_{x_1} < \infty, \exists s \geq H_{x_1}, X_i(s) \leq x] \\ &\quad + \mathbf{P}[X_i(s) \in \{1, \ldots, x_1\},\ \forall s \in [0, t]]. \end{aligned} \tag{29}$$

As in Lemma 4.1, we have

$$\mathbf{P}[X_i(s) \in \{1, \ldots, x_1\}, \forall s \in [0, t+1]] \leq (1 - q_{x_1}(\lambda))\mathbf{P}[X_i(s) \in \{1, \ldots, x_1\}, \forall s \in [0, t]],$$

which implies the existence of a constant $C > 0$ depending only on λ and ϵ' such that

$$\mathbf{P}[X_i(s) \in \{1, \ldots, x_1\},\ \forall s \in [0, t]] \leq e^{-Ct}.$$

Finally, using (28) and (29), there exists some deterministic $t_0 \geq 0$, depending only on λ and ϵ', such that

$$\mathbf{P}[\exists s \geq t, X_i(s) \in \{1, \ldots, x\}] < 2\epsilon',$$

for all $t \geq t_0$ and $\epsilon \leq c_0$.

Besides, if we define σ as the time elapsed between τ_i^e and the first jump time of $\mathcal{N}_2^b$ greater than τ_i^e, then σ is an exponential random variable with parameter ϵ independent of the trajectory of X_i. As a consequence, we have

$$\begin{aligned}\mathbf{P}[\tau_{i+1\to i} < \infty, X_i(\tau_{i+1\to i}) \leq x] &\leq \mathbf{P}[\exists t \geq \sigma, X_i(t) \in \{1, \ldots, x\}], \\ &\leq \mathbf{P}[\exists t \geq \sigma, X_i(t) \in \{1, \ldots, x\}, \sigma > t_0] + \mathbf{P}[\sigma \leq t_0],\end{aligned}$$

$$< 2\epsilon' + 1 - e^{-\epsilon t_0}.$$

Finally we have $\mathbf{P}[\tau_{i+1\to i} < \infty, X_i(\tau_{i+1\to i}) \le x] \underset{\epsilon\to 0}{\longrightarrow} 0$ uniformly in i. □

Now we are able to prove that when a second class particle meets another one, both survive with a probability going to 1 as ϵ goes to 0.

Corollary 4.12

$$\mathbf{P}\left[\tau_{i+1\to i} < \infty, \tau^s_{i+1} < \infty\right] \underset{\epsilon\to 0}{\longrightarrow} 0, \quad \textit{uniformly in } i. \tag{30}$$

Proof Fix $\epsilon' > 0$ and let x_0 be given by Lemma 4.9. We have

$$\begin{aligned}\mathbf{P}\left[\tau_{i+1\to i} < \infty, \tau^s_{i+1} < \infty\right] &= \mathbf{P}\left[\tau_{i+1\to i} < \infty, \tau^s_{i+1} < \infty, X_i(\tau_{i+1\to i}) \le x_0\right]\\ &\quad + \mathbf{P}\left[\tau_{i+1\to i} < \infty, \tau^s_{i+1} < \infty, X_{i+1}(\tau_{i+1\to i}) \ge x_0\right],\\ &\le \mathbf{P}\left[\tau_{i+1\to i} < \infty, X_i(\tau_{i+1\to i}) \le x_0\right]\\ &\quad + \mathbf{P}\left[\tau^s_{i+1} < \infty, \exists t \ge 0, X_{i+1}(t) \ge x_0\right],\\ &< 2\epsilon',\end{aligned}$$

if ϵ is small enough. □

4.2.5 The Proof of Theorem 4.3

Fix $\epsilon' > 0$ and use (30) to find $\epsilon > 0$ small enough to have

$$\mathbf{P}\left[\tau_{i+1\to i} < \infty, \tau^s_{i+1} < \infty\right] < \epsilon'.$$

We have already seen that both N_t/t and N_t^{inf}/t converge to almost sure limits and that $\frac{1}{\epsilon}\lim_{t\to\infty} N_t^{inf}/t$ converges almost surely to $\lambda(1-\lambda)p(\lambda)$ as ϵ goes to 0. Recall the definition of $\bar{N}_t$ at the beginning of the proof of Proposition 4.8. We have $\lim_{t\to\infty}\bar{N}_t/t = \lambda(1-\lambda)\epsilon$. Thus if we define

$$\tau := \inf\left\{t \ge 0 : \forall s \ge t, \frac{\bar{N}_s}{s} \le (\lambda(1-\lambda)+1)\epsilon\right\},$$

then τ is almost surely finite and $N_t^{inf} - N_t = \sum_{i=1}^{\bar{N}_t} \mathbf{1}_{S_i^{inf}(t)=1, S_i(t)=0}$ which implies

$$\begin{aligned}\mathbf{E}\left[\frac{N_t^{inf} - N_t}{t}\mathbf{1}_{\tau\le t}\right] &\le \frac{1}{t}\sum_{i=1}^{\lfloor(\lambda(1-\lambda)+1)\epsilon t\rfloor} \mathbf{P}\left[S_i^{inf}(t) = 1, S_i(t) = 0, \tau \le t\right],\\ &\le \frac{1}{t}\sum_{i=2}^{\lfloor(\lambda(1-\lambda)+1)\epsilon t\rfloor+1} \mathbf{P}\left[\tau_{i\to i-1} < \infty, \tau^s_i < \infty\right],\\ &\le \frac{\lfloor(\lambda(1-\lambda)+1)\epsilon t\rfloor + 1}{t}\epsilon',\end{aligned}$$

and, by dominated convergence theorem, the left-hand side of the above inequality converges to $\lim_{t\to\infty} N_t^{inf}/t - \lim_{t\to\infty} N_t/t$ as t goes to infinity. Hence, dividing by ϵ, we get

$$0 \le \frac{1}{\epsilon}\lim_{t\to\infty}\frac{N_t^{inf}}{t} - \frac{1}{\epsilon}\lim_{t\to\infty}\frac{N_t}{t} \le (\lambda(1-\lambda)+1)\epsilon'.$$

Since ϵ' was arbitrary we can conclude.

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